Optimal Gearing

Not all long-short portfolios are efficient.

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With the developments in quantitative finance over the past 50 years, investors have increasingly understood the importance of engineering portfolios to represent their views effectively. Markowitz [1959] began this process in the 1950s by specifying a clear mathematical approach to trading off expected return and risk. Sharpe [1963], Treynor and Black [1973], Rosenberg [1976], and others advanced the process by focusing active investors on how to build portfolios to beat specific performance benchmarks.

More recent work by Grinold and Kahn [2000a] and Clarke, de Silva, and Thorley [2002] has proposed specific measures of implementation efficiency—especially the transfer coefficient—to capture precisely how well portfolios represent underlying views. This work has improved our understanding of how constraints and costs impact portfolios.

One particular application has been an analysis of the impact of the long-only constraint, quantifying the advantages of long-short investing. Clarke, de Silva, and Sapra [2004] have further analyzed the advantages of partial long-short investing.

Most investors today understand that long-short portfolios reflect manager views more accurately than long-only portfolios. But is that the end of the story? Do all long-short portfolios exhibit similar efficiencies? Or, do even long-short portfolio managers face choices that can significantly impact their efficiency?

This article describes another important dimension—gearing—along which to consider implementation efficiency. Consider an equity market-neutral portfolio manager designing a $100 million product to deliver the maximum possible expected alpha at 10% risk. She can

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invest \$100 million in a 10% risk portfolio. Or, she can use \$100 million to collateralize a \$300 million portfolio run at 3.3% risk. Or she can choose among a continuum of possible combinations of active risk and leverage.

As we will show, portfolio efficiency and the expected alpha of the portfolio vary in surprising, interesting, and important ways along that continuum. There is an optimal level of gearing to maximize efficiency, but, conversely, a poor choice of gearing can make a poor long-short implementation even less efficient than long-only.

We introduce the method using a simple model and embellishments. We provide both a rule of thumb to judge whether a portfolio is overgeared or undergeared and an analysis of an actual portfolio facing exactly this issue.

SIMPLE MODEL

We first find the optimal relation between gearing and risk in a simple model, using the general framework of Grinold and Kahn [2000b]. Later, we generalize this model to investigate the efficiency loss due to incompatible gearing and risk levels.

We start with an investment universe consisting of N stocks, with uncorrelated residual returns, θ_n, and identical residual risk levels, ω_0. We assume individual stock expected residual returns have the form:

$$E\{\theta_n\} \equiv \alpha_n = IC \cdot \omega_0 \cdot z_n \tag{1}$$

with the same information coefficient, IC, for each stock. The terms $\{z_n\}$ are independent, normally distributed, random variables with mean 0 and standard deviation 1.

Combining Equation (1) with the active management utility function:

$$\begin{aligned} U &= \mathbf{h}^T \cdot \boldsymbol{\alpha} - \lambda \mathbf{h}^T \cdot \mathbf{VR} \cdot \mathbf{h} \\ &\Rightarrow \sum_{n=1}^{N} h_n \cdot \alpha_n - \lambda h_n^2 \cdot \omega_0^2 \end{aligned} \tag{2}$$

where $\mathbf{h}$ represents portfolio holdings, and $\mathbf{VR}$ is the covariance matrix of residual returns, leads to optimal holdings:

$$h_n^* = \frac{\alpha_n}{2\lambda\omega_0^2} \Rightarrow \left(\frac{IC}{2\lambda}\right) \cdot \left(\frac{z_n}{\omega_0}\right) \tag{3}$$

Using Equation (3), we can calculate the portfolio alpha and risk as

$$\begin{aligned} \alpha^* &= \left(\frac{IC^2}{2\lambda}\right) \cdot \sum_{n=1}^{N} z_n^2 \\ \omega^* &= \left(\frac{IC}{2\lambda}\right) \cdot \sqrt{\sum_{n=1}^{N} z_n^2} \end{aligned} \tag{4}$$

Now where does gearing enter here? We will define as *unlevered* a market-neutral portfolio that is 100% long and 100% short, with 100% deposited in a collateral account. For example, an unlevered \$100 million market-neutral portfolio would have \$100 million long, \$100 million short, and \$100 million deposited in a collateral account.

While this is our definition in this article—and it is a common definition—others define this as two-times levered. With our view, we more generally define portfolio leverage, or *gearing*, as:

$$G \equiv \frac{1}{2} \sum_{n=1}^{N} | h_n | \tag{5}$$

According to Equation (5), a portfolio 100% long and 100% short has a gearing of one. In our simple model the portfolio gearing is related to the z-scores by

$$G^* = \frac{1}{2}\left(\frac{IC}{2\lambda\omega_0}\right) \cdot \sum_{n=1}^{N} | z_n | \tag{6}$$

If N is reasonably high, we can replace the sums in (4) and (6) by their expectations. Using $E\{z^2\} = 1$ and $E\{|z|\} = \sqrt{\frac{2}{\pi}}$, we find:

$$\begin{aligned} \alpha^* &= \frac{IC^2 \cdot N}{2\lambda} \\ \omega^* &= \frac{IC \cdot \sqrt{N}}{2\lambda} \\ G^* &= \frac{IC \cdot N}{\sqrt{2\pi}\,\lambda\omega_0} \end{aligned} \tag{7}$$

A critical point is that there is only a single degree of freedom in Equation (7); if the investor fixes the risk aversion, that in turn fixes *all three* of the expected alpha, risk, and gearing. If the investor fixes the expected alpha instead, that determines the risk aversion, risk, and gearing. Attempting to specify any two of these independently runs the risk of pushing the portfolio away from the optimal relations among the expected alpha, risk, and gearing, with negative consequences for the *IR*.

In practice, typically investors fix the risk level. Eliminating the risk aversion in favor of the risk, we find a relation between risk and gearing at optimality:

$$G^{*} = \left(\frac{\omega^{*}}{\omega_0}\right) \cdot \sqrt{\frac{N}{2\pi}} \tag{8}$$

What can we learn from Equation (8)? First, at a high level, it relates the optimal gearing to the ratio of target portfolio residual risk, ω^*, to $\omega_0 / \sqrt{N}$, the residual risk of an equal-weighted portfolio of N stocks, each with residual risk, ω_0. The higher that ratio, the higher the optimal gearing.

Consider an example. Our investment universe comprises 250 stocks, each with 25% residual volatility, and we build an optimal portfolio with 5% risk. According to Equation (8), that portfolio has a natural gearing of 1.26. That is, a completely unconstrained portfolio would naturally end up with gearing of 1.26. We can also use Equation (8) to show that an unlevered portfolio ($G = 1$) has a natural volatility of 3.96%.

Now let's consider two different situations mentioned earlier. What if our portfolio manager builds a 10% risk portfolio with a gearing of one? Forcing the gearing to be exactly one, i.e., imposing a gearing constraint, leads to a suboptimal portfolio. The natural ungeared portfolio has 3.96% risk. To increase that to 10%, while keeping gearing fixed, requires dropping assets to reach that higher risk level. At the extreme, the portfolio could end up with only one stock long and one stock short. In these undergeared portfolios, we are giving up on diversification, and on our information ratio, in order to reach a target risk level.

The second situation involves building a 3.3% risk portfolio and gearing it three times. That involves overgearing. Our 3.3% portfolio will need to ignore some alpha information to reach that artificially low risk level. At the extreme, we will build equally weighted long and short portfolios (given that each stock has identical and independent residual risks). This solution, forcing the gearing of three, also leads to suboptimal portfolios.

SIMPLE MODEL EXTENDED: GEARING PENALTIES

Of course in the real world, portfolios may not have an optimal trade-off between risk and gearing. Suboptimal gearing has a cost.

To analyze its impact, we can extend the simple model's utility function, Equation (2), to include a penalty for gearing. To do this, we will account separately for long positions, l_n, and short positions, s_n, defining l_n and s_n as non-negative. In this simple model with uncorrelated residual returns, no transaction costs, and symmetric alphas, a positive alpha will on average result in $s_n = 0$, and a negative alpha will on average result in $l_n = 0$. In what follows we will enforce that explicitly.

Our extended utility function is:

$$\begin{aligned} U \Rightarrow & \sum_{n=1}^{N} \alpha_n \cdot (l_n - s_n) - \lambda \cdot \omega_0^2 \cdot \sum_{n=1}^{N} (l_n - s_n)^2 \\ & - IC \cdot \omega_0 \cdot \phi \cdot \sum_{n=1}^{N} l_n - IC \cdot \omega_0 \cdot \phi \cdot \sum_{n=1}^{N} s_n \\ & - \sum_{n=1}^{N} \eta_n^l \cdot l_n - \sum_{n=1}^{N} \eta_n^s \cdot s_n \end{aligned} \tag{9}$$

The first two terms in Equation (9) represent the alpha and risk as before. Remember that, depending on the sign of the alpha, we know that for any particular asset either l_n or s_n is zero. The (new) third and fourth terms penalize or encourage gearing, depending on whether ϕ, the gearing penalty, is positive or negative. We multiply these terms by $IC \cdot \omega_0$ to ease the notational burden later. As a beneficial side effect, this also gives the gearing penalty a natural scale; for instance, $\phi = 1$ creates a gearing penalty equal to a one-standard deviation alpha. The last two terms—asset-specific holdings penalties—provide degrees of freedom necessary to satisfy the basic conditions that long and short holdings are non-negative.

Note that the gearing penalty doesn't force our solution to a specific gearing. The penalty approach can do this on average, but different samples of alphas lead to different gearing levels that vary around an average.

Our game plan is to solve for the optimal holdings, and see how our alpha, risk, gearing, and information ratio change as we vary the gearing penalty. We present the results here; further details of the calculations are in Appendix A.

For the portfolio alpha, α_P, we find:

$$\frac{\alpha_P}{\alpha^*} = \sqrt{\frac{2}{\pi}} \cdot [I_2(\phi_+) - \phi I_1(\phi_+)] \tag{10}$$

where

$$I_n(x) \equiv \int_x^\infty \exp\left\{\frac{-y^2}{2}\right\} y^n \, dy \tag{11}$$

and

$$\phi_+ \equiv \begin{cases} 0 & \phi < 0 \\ \phi & \phi \geq 0 \end{cases} \tag{12}$$

The quantity α^* is defined in Equation (7), and is the alpha of the gearing-unconstrained ($\phi = 0$) portfolio.

Similarly, we can estimate portfolio risk, ω_p, and gearing as:

$$\frac{\omega_P}{\omega^*} = \left(\frac{2}{\pi}\right)^{1/4} \sqrt{I_2(\phi_+) - 2\phi I_1(\phi_+) + \phi^2 I_0(\phi_+)} \tag{13}$$

$$\frac{G}{G^*} = I_1(\phi_+) - \phi I_0(\phi_+) \tag{14}$$

where, again, we computed ω^* and G^* in Equation (7).

The intrinsic information ratio (*IR*) follows from the fundamental law of active management (Grinold [1989]), and is:

$$IR_{int} = \frac{\alpha^*}{\omega^*} = IC \cdot \sqrt{N} \tag{15}$$

The implemented information ratio is simply the ratio of the alpha and risk estimates. From these, we can calculate the transfer coefficient, *TC*, as the ratio of implemented information ratio to intrinsic information ratio:

$$TC = \frac{\left(\alpha_P / \omega_P\right)}{\left(\alpha^* / \omega^*\right)} = \frac{IR}{IC \cdot \sqrt{N}}$$

$$= \left(\frac{2}{\pi}\right)^{1/4} \left[\frac{I_2(\phi_+) - \phi I_1(\phi_+)}{\sqrt{I_2(\phi_+) - 2\phi I_1(\phi_+) + \phi^2 I_0(\phi_+)}}\right] \tag{16}$$

Notice that, unlike the *IR*, the transfer coefficient is independent of *N*. With a gearing penalty of zero, the optimization naturally chooses the optimal gearing, and the transfer coefficient is 1.

Exhibit 1 shows the transfer coefficient at a fixed gearing, and Exhibit 2 shows the transfer coefficient at a fixed risk.

CHARACTERISTICS OF UNDERGEARED VERSUS OVERGEARED PORTFOLIOS

Interestingly, undergeared and overgeared portfolios are quite different, and this leads to an asymmetry in the transfer coefficient for deviations away from optimal gearing. In fact, undergeared and overgeared portfolios are so different that it is possible to distinguish between severely undergeared and overgeared portfolios with only a glance at the portfolio holdings.

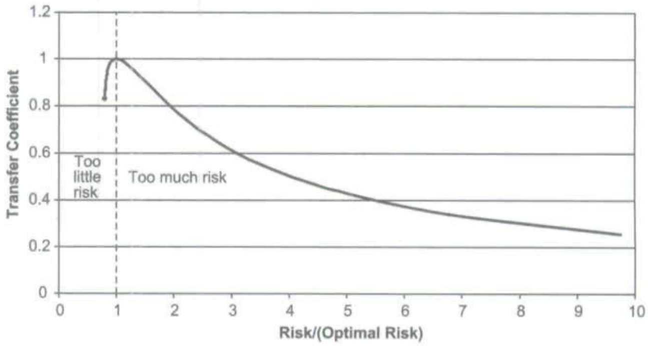
Exhibit 1 shows how the transfer coefficient varies as a function of risk, with the portfolio held at fixed gearing throughout. This graph is derived from Equations (13), (14), and (16) by eliminating the gearing penalty ϕ in terms of the gearing level and the risk aversion λ in terms of the risk level.

In the original portfolio manager example, we used Equation (8) to show that if the manager is constrained to $G = 1$, the ideal risk level is 3.96%. In Exhibit 1, the horizontal axis is measured in multiples of the ideal risk, and for a multiple of 1, the transfer coefficient is at its maximum of 1. In this simple model, changes in the target gearing G would change the optimal risk level but not the shape of Exhibit 1.

At the right-hand side of Exhibit 1, the portfolio bears too much risk for the given level of gearing. To

EXHIBIT 1

Transfer Coefficient at Fixed Gearing



increase the risk, more and more portfolio weight is concentrated in high-alpha names, and assets with low alphas are forced out of the portfolio entirely (thereby discarding all the information in those alphas). In the limiting case, we end up with a single name on the long side and a single name on the short side. This may be fine from an alpha perspective but it is devastating from a risk perspective, and the huge increase in risk causes the transfer coefficient to drop to zero in our simple model. More plausibly, a portfolio with three times the ideal risk level has a transfer coefficient of only about 0.6, the same ballpark as an otherwise unconstrained long-only implementation.

Another perspective is that these high-risk, concentrated portfolios are undergeared, a point we return to below. A key characteristic of an undergeared portfolio is the collection of assets at exactly zero weight; this characteristic persists even in more realistic optimizations that include bounds, transaction costs, and so on.

At the left-hand side of Exhibit 1, the portfolio experiences too little risk for the level of gearing. The transfer coefficient drops off more quickly, but it proves impossible to reduce the risk beyond a certain point, and at that point the transfer coefficient is still over 80% of the maximum. The lower bound on the risk is a consequence of Hölder's Inequality, which relates the risk level and the gearing level (see Appendix B). As we force the portfolio to lower and lower risk levels, it overdiversifies by taking substantial positions even in assets with low alphas. In the limiting case, the portfolio consists of two equal-weighted portfolios: All the assets with positive alphas are held with equal weight on the long side, and all the assets with negative alphas are held with equal weight on the short side. The portfolio ignores all but the sign of the alpha, but because there is still information in the sign alone, the *IR* approaches a positive constant as the portfolio approaches the minimum risk.*

Again, another perspective is that these low-risk portfolios are overgeared, and, given the earlier interpretation, it's clear that an identifying characteristic of an overgeared portfolio is the absence of assets in a region around zero weight, and again, this characteristic persists even in more realistic optimizations.

Exhibit 2 displays the transfer coefficient as a function of gearing, for fixed risk. On the right, the portfolios are overgeared, and tend to have equal-weighted long and short sides, with no assets near zero weight. On the left, the portfolios are undergeared, and tend to have a collection of assets at exactly zero weight. The transfer coefficient drops off more rapidly for overgeared portfolios, but is bounded below, while severe undergearing can reduce the transfer coefficient all the way to zero.

To summarize the differences, undergeared portfolios tend to show an abnormally large number of holdings at exactly zero weight; overgeared portfolios tend to show an abnormal absence of assets in an entire region around zero weight; and perfectly geared portfolios show a smooth distribution of portfolio weights. In our simple model with no transaction costs and normally distributed alphas, the portfolio weights [Equation (3)] are normally distributed when the portfolio is optimally geared.

For completeness, Exhibit 3 shows the efficient frontier for three different levels of gearing, as well as the gearing unconstrained line. The form of the efficient frontiers follows from the analysis so far. There is a point of

EXHIBIT 2
Transfer Coefficient at Fixed Risk

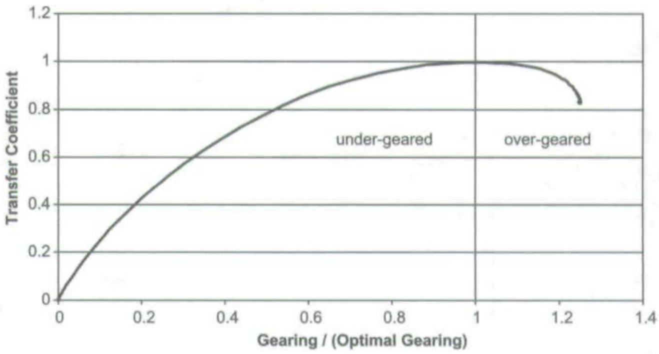
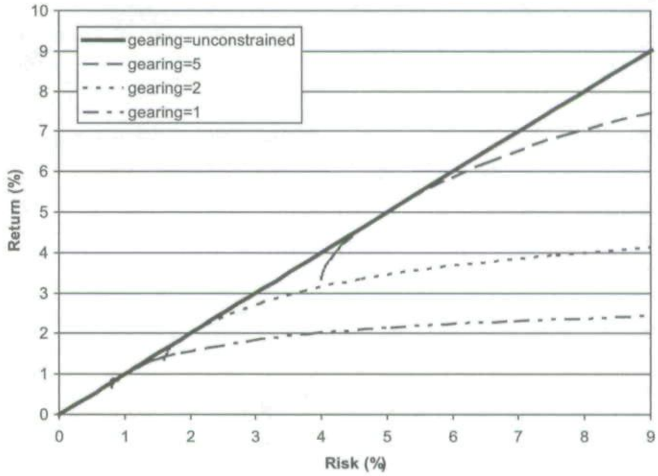


EXHIBIT 3
Efficient Frontiers



highest *IR*, where the gearing, expected risk, and expected return are all compatible. To the left of that point, the portfolio is overgeared, and the efficient frontier terminates at the maximally overgeared portfolio, in this case an equal-weighted portfolio. To the right of the point of maximum *IR*, the portfolios are undergeared. Although our simple model shows no upper bound on the risk, in principle at the far right the efficient frontier also terminates, at the risk and return characteristics of the maximum alpha asset.

Exhibit 3 also clarifies that the scenarios we are considering here aren't simply movements along the capital market line. The capital market line is straight precisely because gearing varies with risk as we move along it. If we fix the gearing, the efficient frontier curves as shown in Exhibit 3.

EMPIRICAL ANALYSIS

We now want to verify aspects of this analysis under more realistic conditions. We can show how gearing affects implemented information ratios by generating random alphas, and building optimal market-neutral portfolios at different risk and gearing levels. For this case, we ignore all constraints other than the gearing constraint, and ignore transaction costs, but include a realistic covariance between assets.

Exhibit 4 shows the subsequent active efficient frontiers based on the expected risk and return. In spite of widely differing underlying parameters, the behavior displayed in Exhibit 4 very closely matches the pattern in Exhibit 3.

The distance between the straight line and the curved frontier is the loss in *IR* from the gearing constraint, and, as we saw before, this can be very high. To achieve levels of risk beyond the optimal point, the portfolio reduces diversification and concentrates in few names. At risk levels below optimal, the portfolio overdiversifies, ignoring information available in the alphas to reduce risk.

Exhibit 5 shows the results of more realistic analysis, including alpha, risk, transaction costs, asset-level bounds, and in three of the four cases a gearing constraint. The graph is again based on expected alpha and risk, and the risk is allowed to go to extremely high levels only to indicate that even the gearing-unconstrained curve dips for high enough risk levels.

A crucial difference between Exhibits 4 and 5 is that in Exhibit 5 the gearing-constrained portfolios have negative expected returns for some risk levels. This is due entirely to expected transaction costs, indicating that the

EXHIBIT 4
Efficient Frontiers—No Other Constraints

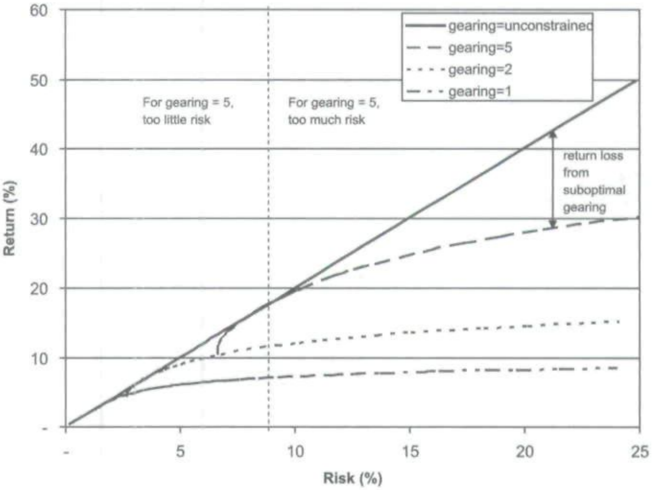
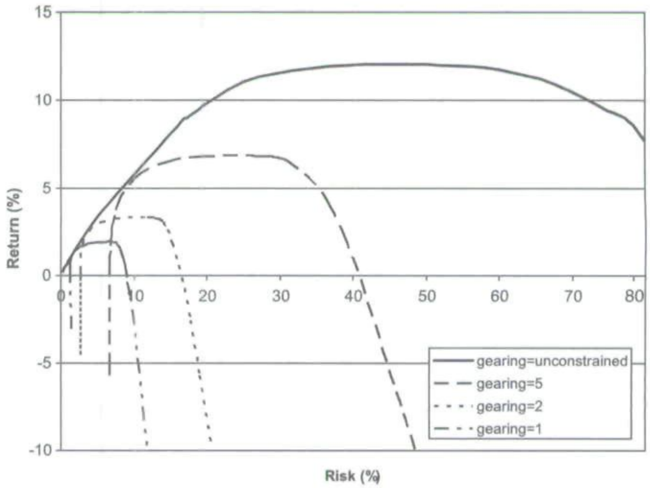


EXHIBIT 5
In Practice Efficient Frontiers



cost of suboptimal gearing is potentially much higher for portfolios that bear significant transaction costs.

In Exhibit 5 the gearing-unconstrained curve has the highest return for any value of the risk, because the portfolio is allowed to naturally find the level of gearing compatible with the risk. The other curves have a shape similar to those generated by our extended model, but have somewhat different behavior at very low and very high risk.

In our simplified model, without transaction costs, the minimum-risk portfolio at a fixed gearing is an equal-weighted portfolio. The minimum-risk portfolios in Exhibit 5 are very similar to those in our simple model, but when the portfolios are subject to transaction costs, it is costly for assets to flip from the long side to the short side or vice versa. The net result is that for minimum-risk portfolios and the level of transaction costs we have chosen, the transaction costs overcome the alpha, leading to a negative expected *IR*. In the middle of each fixed-gearing curve, there is a more normal balance among alpha, risk, transaction costs, and gearing, resulting in *IR*s that are positive, but still lower than the gearing-unconstrained *IR*. As before, at a fixed gearing there is a single optimal risk level, or at a fixed risk there is a single optimal gearing level. Finally, to the far right in Exhibit 5, at very high risk levels the performance drops precipitously again as excessive transaction costs are incurred in order to reach the risk target.

OTHER REAL-WORLD ISSUES

Empirical analysis inevitably identifies other practical issues important for determining optimal gearing. For instance, optimal gearing will vary over time. From Equation (8), we know that gearing depends on the risk target, the asset risk levels, and number of the assets in the opportunity set. As asset risk levels change, to keep portfolio risk constant and the transfer coefficient high requires changing gearing in a compatible way. As volume patterns change, or funds grow, even N can change as less liquid names move out of the investment universe.

Our models so far have assumed stock borrowing is free, which doesn't hold in the real world. If analysis implies that we should increase leverage, for example, and run a portfolio at lower risk, we must also account for the higher stock borrowing costs that will entail.

Finally, the regulatory environment may limit the allowed values for the gearing, even to the point of eliminating the point of highest *IR*. In particular, if regulations force the portfolios to be undergeared (as opposed to overgeared), the resulting transfer coefficients can potentially be as low as the transfer coefficients of long-only portfolios.

CONCLUSIONS

In constructing investment products, managers may set risk targets according to client preferences, and meet those risk targets by altering the asset mix suboptimally. In products that use leverage, managers typically set gearing independent of choice of risk. Yet, as we have shown, risk and gearing incompatibility have a surprisingly high cost. This efficiency loss is asymmetric; undergearing is ultimately more costly than overgearing. Overgeared portfolios are identified by their equal weighting of assets, with no assets near absolute weight of zero; undergeared portfolios are identified by an abnormal collection of assets at exactly zero weight.

While this analysis has focused on the costs of ignoring the connection between risk and gearing in market-neutral portfolios, it is but an example of a more general issue beyond market neutral—compatibly setting portfolio risk, gearing, and shorting when we can freely specify all three.

APPENDIX A

Portfolios Subject To Gearing Penalties

Maximizing utility leads to the holdings:

$$l_n = \frac{1}{2\lambda\omega_0^2}(\alpha_n - IC\cdot\omega_0\cdot\phi - \eta_n^l)$$
$$s_n = \frac{1}{2\lambda\omega_0^2}(-\alpha_n - IC\cdot\omega_0\cdot\phi - \eta_n^s) \tag{A-1}$$

where we choose η_n^l and η_n^s to enforce $l_n \ge 0$ and $s_n \ge 0$. For instance, if $(\alpha_n - IC\cdot\omega_0\cdot\phi) \le 0$ for a positive alpha, we set η_n^l to exactly $(\alpha_n - IC\cdot\omega_0\cdot\phi)$, to pin the asset at $l_n = 0$. If $(\alpha_n - IC\cdot\omega_0\cdot\phi) \ge 0$ for a positive alpha, we need not enforce $l_n \ge 0$ explicitly, and can set η_n^l to zero.

Given the optimal portfolio, Equation (A-1), we can calculate its characteristics—alpha, residual risk, and gearing—and how they vary as we change the risk aversion and the gearing penalty.

While we must separately analyze the cases of positive and negative penalties, ϕ, for gearing, we can summarize the results with single formulas. To calculate the expected portfolio alpha, we estimate the expected average contribution per stock (active position times stock alpha), and multiply by the number of stocks.

Three integrals arise in the calculations below. They are all of the type described in Equation (11). Some of these integrals can be done explicitly. A few particular values of use are: $I_0(0) = I_2(0) = \sqrt{\frac{\pi}{2}}$ and $I_1(0) = 1$.

We will calculate the portfolio risk in detail, starting with the simpler $\phi \le 0$ case. The short and long sides give the same contribution, so we need to calculate only one. The expected variance of an asset on the long side is:

$$\omega_0^2 E[l_i^2] = \omega_0^2 \int_0^\infty P(\alpha)\, l^2 \, d\alpha$$

$$= \omega_0^2 \int_0^\infty \frac{1}{\sqrt{2\pi IC^2\omega_0^2}} \exp\left\{\frac{-\alpha^2}{2IC^2\omega_0^2}\right\}\left(\frac{1}{2\lambda\omega_0^2}\right)^2 \times (\alpha - IC\cdot\omega_0\cdot\phi)^2 \, d\alpha$$

$$= \frac{IC^2}{4\lambda^2\sqrt{2\pi}} \int_0^\infty \exp\left\{\frac{-x^2}{2}\right\}(x-\phi)^2 \, dx$$

$$= \frac{IC^2}{4\lambda^2\sqrt{2\pi}}[I_2(0) - 2\phi I_1(0) + \phi^2 I_0(0)] \tag{A-2}$$

When $\phi \geq 0$, the gearing penalty forces the long side assets toward zero, and if the alpha is low enough the gearing penalty will force the asset out of the portfolio entirely. Then:

$$\omega_0^2 E[l_i^2] = \omega_0^2 \int_0^\infty P(\alpha)\, l^2 \, d\alpha$$

$$= \omega_0^2 \int_0^\infty \frac{1}{\sqrt{2\pi\omega_0^2}} \exp\left\{\frac{-\alpha^2}{2IC^2\omega_0^2}\right\}\left(\frac{1}{2\lambda\omega_0^2}\right)^2 \times (\alpha - IC\cdot\omega_0\cdot\phi)_+^2 \, d\alpha$$

$$= \frac{IC^2}{4\lambda^2\sqrt{2\pi}} \int_\phi^\infty \exp\left\{\frac{-x^2}{2}\right\}(x-\phi)^2 \, dx$$

$$= \frac{IC^2}{4\lambda^2\sqrt{2\pi}}[I_2(\phi) - 2\phi I_1(\phi) + \phi I_0(\phi)] \tag{A-3}$$

The $\phi \geq 0$ and $\phi \leq 0$ cases together can be written as:

$$\omega_0^2 E[l_i^2] = \frac{IC^2}{4\lambda^2\sqrt{2\pi}}[I_2(\phi_+) - 2\phi I_1(\phi_+) + \phi^2 I_0(\phi_+)] \tag{A-4}$$

Recalling that both the long and short sides contribute to the risk, the total portfolio risk is

$$\frac{\omega_P}{\omega^*} = \left(\frac{2}{\pi}\right)^{1/4} \sqrt{I_2(\phi_+) - 2\phi I_1(\phi_+) + \phi^2 I_0(\phi_+)} \tag{A-5}$$

in terms of the $\phi = 0$ risk level, $\omega^* = \frac{IC\sqrt{N}}{2\lambda}$.

The expected gearing and alpha calculations are very similar to the expected variance calculation, and so are omitted. The formula for the gearing is

$$\frac{G}{G^*} = I_1(\phi_+) - \phi I_0(\phi_+) \tag{A-6}$$

in terms of the $\phi = 0$ gearing:

$$G^* = \frac{N\cdot IC}{2\lambda\omega_0\sqrt{2\pi}}$$

The formula for the alpha is

$$\frac{\alpha_P}{\alpha^*} = \sqrt{\frac{2}{\pi}}[I_2(\phi_+) - \phi I_1(\phi_+)] \tag{A-7}$$

in terms of the $\phi = 0$ alpha:

$$\alpha^* = \frac{N\cdot IC^2}{2\lambda}.$$

APPENDIX B

Hölder's Inequality and the Upper Bound on Gearing

Hölder's inequality is

$$\sum_{i=1}^N |x_i y_i| \leq \left(\sum_{i=1}^N |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^N |y_i|^q\right)^{\frac{1}{q}} \tag{B-1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Plugging $p = q = 2$, $x_i = \frac{1}{\omega_i}$, and $y_i = \omega_i h_i$ into Hölder's inequality yields

$$\sum_{i=1}^N |h_i| \leq \sqrt{\sum_{i=1}^N \frac{1}{\omega_i^2}} \sqrt{\sum_{i=1}^N \omega_i^2 h_i^2}$$

or

$$G \leq \frac{\sqrt{N}}{2} \frac{\omega^*}{\overline{\omega}} \tag{B-2}$$

where ω_i is the specific risk of asset i, ω^* is the portfolio specific risk, and $\overline{\omega}$ is a typical asset risk, defined by

$$\overline{\omega} \equiv \left(\frac{1}{N}\sum_i {}^1\!/_{\omega_i^2}\right)^{-1/2}$$

Combining the upper bound with the calculation of optimal gearing in our simple model shows that the ratio of the maximum gearing to the optimal gearing satisfies:

$$\frac{G_{\max}}{G^*} = \sqrt{\frac{\pi}{2}} \approx 1.25 \tag{B-3}$$

This ratio is a pure number, independent of the risk level, *IC*, and so on. Notice that the maximum gearing produced by our extended simple model satisfies this relation.

Finally, note that similar bounds can be found by applying the Cauchy-Schwarz inequality, which is obviously related to Hölder's inequality.

ENDNOTE

*The equal-weighting follows from our assumption that all assets have equal risk and are uncorrelated. More generally, overgearing should, in the limit, lead to long and short portfolios built to minimize risk.

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